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Compatibility of $1/n$ and ϵ expansions for critical exponents at m-axial Lifshitz points

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Abstract
The critical behaviour of d-dimensional n-vector models at m-axial Lifshitz points is considered for general values of m in the large-n limit. It is proven that the recently obtained large-n expansions (Shpot *et al* 2005 *J. Phys.: Condens. Matter* **17** S1947) of the correlation exponents η_{L2}, η_{L4} and the related anisotropy exponent θ are fully consistent with the dimensionality expansions to second order in $\epsilon = 4 + m/2 - d$ (Diehl and Shpot 2000 *Phys. Rev.* B **62** 12338; Shpot and Diehl 2001 *Nucl. Phys.* B **612** 340) inasmuch as both expansions yield the same contributions of order ϵ^2/n.

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1. Introduction

Lifshitz points (LPs) are familiar examples of multi-critical points. At a LP a disordered, a homogenous ordered, and a modulated ordered phase meet [1–4]. In the case of systems with m-axial LPs, there is a degeneracy such that long-range order modulated along any of m distinct axes can occur in the modulated ordered phases. The study of critical behaviour at such LPs began immediately after their discovery in the middle of the 1970s.

Unfortunately, the technical difficulties one is faced with in analytical renormalization group (RG) calculations are enormous. This is the main reason why RG results based on systematic expansions, such as expansions in powers of $\epsilon = d^*(m) - d$ about the upper critical dimension $d^*(m) = 4 + m/2$, or in powers of $1/n$, where n is the number of components of the order parameter, had remained quite scarce for decades. Furthermore, early ϵ-expansion results obtained by two different groups (cf [5–7]) had yielded contradictory results, and these discrepancies had remained unclarified for many years. The results of [5], which were restricted to the special cases of bi- and hexa-axial LP $m = 2$ and $m = 6$, were reproduced 20 years later by field-theoretic means [8]. However, a full two-loop RG analysis in $d^*(m) - \epsilon$ dimensions was reported only in 2001 [9, 10]. This gave the ϵ expansions to second order of

all four main independent critical exponents as well as the correction-to-scaling exponent for general values of m, besides resolving the mentioned discrepancies[4].

In a recent paper [13] (hereafter referred to as I), we have shown how the $1/n$ expansion can be applied to the study of critical behaviour at m-axial LPs. We determined the correlation exponents η_{L2} and η_{L4}, and the related anisotropy index θ of d-dimensional systems to first order in $1/n$ for general values of m. The results took the form of complicated integrals whose integrands involve further multi-dimensional integrals. We were able to check that in the isotropic limits $m \to 0$ and $m \to d$ they correctly reduce to known results, namely, the expansions to order $1/n$ of the Fisher exponent η at a usual critical point [14, 15], and that of η_{L4} at the isotropic d-axial LP [16], respectively. Furthermore, we could show that our $O(1/n)$ results are in conformity with published dimensionality expansion results about the lower critical dimension $d_*(m) = 2+m/2$ both for $m = 0$ [17] and for $m = 2$ and $m = 6$ [18].

The virtues of the $1/n$ expansion are well known: it can be applied in arbitrary fixed dimensions d, does not rely on the smallness of a further expansion parameter such as ϵ and yields nontrivial results below the upper critical dimension $d^*(m)$ in a mathematically controlled fashion. Besides its capability of providing valuable information about the critical behaviour for given d and m, it allows for nontrivial checks on ϵ-expansion results. Unfortunately, the complicated general form of our results in I prevented us from proving their consistency with the ϵ-expansion results of [9, 10] for general m. We could verify it in the isotropic limits $m \to 0$ and $m \to d$. However, the only other case in which we could explicitly demonstrate this consistency by analytical means was that of $m = 2$. This is unfortunate for at least two reasons: first, it excludes, in particular, the physically important uniaxial case $m = 1$; second, unlike both the contributions of first order in $\epsilon = d^*(m) - d$ as well as their $\mathrm{O}(d - d_*(m))$ analogues in the expansions about the lower critical dimensions $d_*(m)$, the $\mathrm{O}(\epsilon^2)$ terms exhibit a nontrivial m-dependence [9, 10]. This propagates into the $\mathrm{O}(1/n)$ series coefficients, and ought to be checked.

The purpose of this paper is to fill this gap and prove that our large-n expansion results [13] to order $1/n$ for the exponents $\eta_{L2}(d, m, n)$, $\eta_{L4}(d, m, n)$ and $\theta(d, m, n)$ are fully consistent with the ϵ-expansion results of [9, 10]. In the following section we first provide the necessary background, recalling the continuum model on which our analysis is based as well as our results for general m given in I. We then show that these results can be rewritten in a form allowing analytic comparisons with the $\mathrm{O}(\epsilon^2)$ results of [9, 10] for general m. The proof that they are in conformity with the latter is given in section 3. The closing section 4 contains a brief discussion and concluding remarks.

2. Large-n expansions of the correlation exponents at the Lifshitz point

Just as in I, we consider a model defined by the Euclidean action

$$\mathcal{H}[\phi] = \frac{1}{2}\int \mathrm{d}^{d-m}r \int \mathrm{d}^m z \left[(\nabla_r \phi)^2 + \left(\nabla_z^2 \phi\right)^2 + \tau_{\mathrm{LP}}\phi^2 + \rho_{\mathrm{LP}}(\nabla_z \phi)^2 + \frac{\lambda}{8}\phi^4 \right]. \tag{1}$$

Since we intend to work directly at the LP, we have set the coefficients of the quadratic terms to their corresponding critical values τ_{LP} and ρ_{LP} [5].

Here $\phi = \phi(\boldsymbol{x})$ is the usual n-component order-parameter field. Its d-dimensional position vector $\boldsymbol{x} = (\boldsymbol{r}, \boldsymbol{z}) \in \mathbb{R}^d$ has a $(d - m)$-dimensional 'perpendicular' component $\boldsymbol{r}$

[4] Alternative results reported in [11] could be refuted [12].

[5] In the following, it is tacitly understood that d with $d < d^*(m) = 4 + m/2$ and m are chosen such that a LP exists. This requires, in particular, that d exceeds the dimension $2 + m/2$ below which the homogeneous ordered phase becomes thermally unstable because of spin-wave excitations. However, it also requires that the modulated ordered phase remains stable. For a discussion of these delicate issues, see the review article [4] and its references.

and an m-dimensional 'parallel' one, $\boldsymbol{z}$. The subspace associated with $\boldsymbol{z}$ is the one in which modulated order can occur in the corresponding phase. The $\boldsymbol{r}$ subspace is its orthogonal complement. A similar decomposition has been made for the gradient operator $\nabla = (\nabla_{\boldsymbol{r}}, \nabla_{\boldsymbol{z}})$. Thus $\nabla_{\boldsymbol{z}}^2$ is the Laplacian in the parallel subspace.

Employing the notational conventions of I, we write the wave vector conjugate to $\boldsymbol{x} = (\boldsymbol{r}, \boldsymbol{z})$ as $\boldsymbol{k} = (\boldsymbol{p}, \boldsymbol{q})$, with $\boldsymbol{p} \in \mathbb{R}^{d-m}$ and $\boldsymbol{q} \in \mathbb{R}^m$. Further, we introduce the two-point cumulant $G_\phi(r, z)$ and its Fourier transform $\tilde{G}_\phi(p, q)$ through

$$\begin{aligned} G_\phi(r, z) &= \frac{1}{n}[\langle\phi(\boldsymbol{r}, \boldsymbol{z}) \cdot \phi(\boldsymbol{0}, \boldsymbol{0})\rangle - \langle\phi(\boldsymbol{r}, \boldsymbol{z})\rangle \cdot \langle\phi(\boldsymbol{0}, \boldsymbol{0})\rangle] \\ &= \int_{\boldsymbol{p}}^{(d-m)} \int_{\boldsymbol{q}}^{(m)} \mathrm{e}^{\mathrm{i}(\boldsymbol{r}\cdot\boldsymbol{p}+\boldsymbol{z}\cdot\boldsymbol{q})} \tilde{G}_\phi(p, q), \end{aligned} \tag{2}$$

where $\int_{\boldsymbol{p}}^{(d-m)} \equiv (2\pi)^{-d+m} \int_{\mathbb{R}^{d-m}} \mathrm{d}^{d-m} p$ and $\int_{\boldsymbol{q}}^{(m)} \equiv (2\pi)^{-m} \int_{\mathbb{R}^m} \mathrm{d}^m q$ denote normalized $(d-m)$- and m-dimensional integrals, respectively.

As discussed in I, the full propagator $\tilde{G}_\phi(p, q)$ becomes a generalized homogeneous function $\tilde{G}_\phi^{(\mathrm{as})}(p, q)$ in the limit of large length scales. The latter function satisfies at the LP the homogeneity relations

$$\tilde{G}_\phi^{(\mathrm{as})}(p, q) = p^{-2+\eta_{L2}} \tilde{G}_\phi^{(\mathrm{as})}(1, qp^{-\theta}) = q^{-4+\eta_{L4}} \tilde{G}_\phi^{(\mathrm{as})}(pq^{-1/\theta}, 1). \tag{3}$$

Only two exponents are independent here since the usual scaling relation

$$\theta = \frac{2 - \eta_{L2}}{4 - \eta_{L4}} \tag{4}$$

must hold for the anisotropy index θ by consistency.

In the limit $n \to \infty$ with $n\lambda =$ fixed, $\tilde{G}_\phi^{(\mathrm{as})}(p, q)$ reduces to the Gaussian propagator $\tilde{G}^{(0)}(p, q)$ pertaining to the Hamiltonian (1) with $\lambda = \tau_{\mathrm{LP}} = \rho_{\mathrm{LP}} = 0$. We have

$$\lim_{\substack{n\to\infty \\ n\lambda=\mathrm{const}}} \tilde{G}_\phi^{(\mathrm{as})}(p, q) = \tilde{G}^{(0)}(p, q) \equiv \frac{1}{p^2 + q^4}. \tag{5}$$

At order $1/n$, self-consistent equations must be solved which were discussed in I and need not be repeated here. To this end, we looked for solutions of the scaling form (3), utilizing the ansätze

$$\eta_{L2} = \frac{\eta_{L2}^{(1)}}{n} + \mathrm{O}(n^{-2}), \qquad \eta_{L4} = \frac{\eta_{L4}^{(1)}}{n} + \mathrm{O}(n^{-2}), \qquad \theta = \frac{1}{2} + \frac{\theta^{(1)}}{n} + \mathrm{O}(n^{-2}), \tag{6}$$

together with the corresponding $1/n$ expansions for the scaling functions in equation (3) and the relation

$$\theta^{(1)} = \frac{\eta_{L4}^{(1)}}{8} - \frac{\eta_{L2}^{(1)}}{4} \tag{7}$$

implied by the scaling law (4). This led to consistency conditions (equations (27) and (28) of I) from which we obtained the results

$$\eta_{L2}^{(1)} = \frac{K_{d-m}}{d-m} \int_{\boldsymbol{q}}^{(m)} \frac{2\mathcal{P}_1(q^4)}{(1+q^4)^3} \frac{1}{I(1, q)} \tag{8}$$

and

$$\eta_{L4}^{(1)} = \frac{K_m}{4m(m+2)} \int_{\boldsymbol{p}}^{(d-m)} \frac{8\mathcal{P}_2(p^2)}{(p^2+1)^5} \frac{1}{I(p, 1)}. \tag{9}$$

Here K_{d-m} and K_m, defined by

$$K_D \equiv \frac{S_D}{(2\pi)^D} \qquad \text{with} \qquad S_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}, \tag{10}$$

where S_D is the area of a unit sphere in D dimensions, are conventional factors resulting from the angular integrations at $D = d - m$ and m. Further, $\mathcal{P}_1(q^4)$ and $\mathcal{P}_2(p^2)$ denote the polynomials

$$\mathcal{P}_1(q^4) = 4 - (d - m)(1 + q^4) \tag{11}$$

and

$$\mathcal{P}_2(p^2) = 3(8 - m)(6 - m) + 5(m^2 + 2m - 96)p^2 + (m^2 + 50m + 144)p^4 - m(m + 2)p^6. \tag{12}$$

Finally, $I(p, q)$ represents the analogue of Ma's 'elementary bubble' [14] for the LP,

$$I(p, q) = \int_{p'}^{(d-m)} \int_{q'}^{(m)} \frac{1}{p'^2 + q'^4} \frac{1}{|\boldsymbol{p}' + \boldsymbol{p}|^2 + |\boldsymbol{q}' + \boldsymbol{q}|^4}, \tag{13}$$

whose homogeneity property

$$I(p, q) = p^{-\epsilon} I(1, qp^{-1/2}) = q^{-2\epsilon} I(pq^{-2}, 1), \qquad \epsilon = 4 - d + m/2, \tag{14}$$

we recall for later use.

Let us first show that the results (8) and (9) for the $\mathrm{O}(1/n)$ coefficients can be rewritten as

$$\eta_{L2}^{(1)} = \frac{K_m}{2(d - m)} \int_{\boldsymbol{p}}^{(d-m)} \frac{1}{p^2 + 1} \nabla_{\boldsymbol{p}}^2 \frac{1}{I(p, 1)} \tag{15}$$

and

$$\eta_{L4}^{(1)} = \frac{K_{d-m}}{2m(m + 2)} \int_{\boldsymbol{q}}^{(m)} \frac{1}{1 + q^4} \nabla_{\boldsymbol{q}}^4 \frac{1}{I(1, q)}, \tag{16}$$

respectively, where $\nabla_{\boldsymbol{q}}^4 \equiv \left(\nabla_{\boldsymbol{q}}^2\right)^2$. These representations of the coefficients $\eta_{L2}^{(1)}$ and $\eta_{L4}^{(1)}$ are well suited for determining the $\mathrm{O}(\epsilon^2/n)$ contributions to the exponents η_{L2} and η_{L4}. They will be employed as the starting point in our proof of consistency with the ϵ-expansion results of [9, 10] given in the following section.

To derive these representations, note first that the actions of the D-dimensional Laplacian $\nabla_{\boldsymbol{K}}^2 = \sum_{\gamma=1}^{D} \partial^2/\partial K_\gamma^2$ and its square $\nabla_{\boldsymbol{K}}^4$ on functions $f(K^2)$ of $K^2 = \sum_{\gamma=1}^{D} K_\gamma^2$ can be written as

$$\nabla_{\boldsymbol{K}}^2 f(K^2) = 2Df'(K^2) + 4K^2 f''(K^2) \tag{17}$$

and

$$\nabla_{\boldsymbol{K}}^4 f(K^2) = 4D(D + 2)f''(K^2) + 16(D + 2)K^2 f^{(3)}(K^2) + 16K^4 f^{(4)}(K^2), \tag{18}$$

where $f^{(s)}(.)$ means the sth derivative of the function $f(.)$.

Using these relations, it is straightforward to see that the rational functions appearing in the integrands of (8) and (9) can be expressed as

$$\frac{2\mathcal{P}_1(q^4)}{(1 + q^4)^3} = \nabla_{\boldsymbol{P}}^2 \left. \frac{1}{P^2 + q^4} \right|_{P^2=1} \tag{19}$$

and

$$\frac{8\mathcal{P}_2(p^2)}{(1 + p^2)^5} = \nabla_{\boldsymbol{Q}}^4 \left. \frac{1}{p^2 + Q^4} \right|_{Q^2=1}. \tag{20}$$

We now insert these results into equations (8) and (9), use hyper-spherical coordinates for the integrals $\int_{\boldsymbol{q}}^{(m)}$ and $\int_{\boldsymbol{p}}^{(d-m)}$, make the changes of variables $q \to p = q^{-2}$ and $p \to q = p^{-1/2}$ in the radial integrals over q and p, and utilize the scaling property (14) to express

$I(1, p^{-1/2})$ and $I(q^{-2}, 1)$ in terms of $I(p, 1)$ and $I(1, q)$ respectively. The derivative term on the right-hand side of equation (19) becomes $p^2\nabla_P^2(P^2p^2+1)^{-1}|_{P=1} = p^2\nabla_p^2(p^2+1)^{-1}$. The ∇_Q^4 term in equation (19) transforms in a corresponding fashion. One thus arrives at expressions that agree with equations (15) and (16) except that the derivatives act to the left. Integration by parts then yields the claimed results.

A straightforward, though important first application of them is to show that the ϵ expansion of the coefficients $\eta_{L2}^{(1)}$ and $\eta_{L4}^{(1)}$ starts at order ϵ^2,

$$\eta_{L2,4}^{(1)} = \eta_{L2,4}^{(1,2)}\epsilon^2 + \mathrm{O}(\epsilon^3). \tag{21}$$

To see this, note that $I(p, q)$ has a Laurent expansion about $\epsilon = 0$ of the form

$$I(p, q) = \frac{I_{-1}}{\epsilon} + I_0(p, q) + \mathrm{O}(\epsilon) \tag{22}$$

with a momentum-independent residuum I_{-1}, given by

$$I_{-1} = (4\pi)^{-(8+m)/4}\frac{\Gamma(m/4)}{\Gamma(m/2)} \tag{23}$$

according to [9, 10] (see equations (7), (24) and (89) of [9] or (38) and (39) of [10], where I_{-1} was denoted $F_{m,0}$). Hence

$$\nabla_p^2\frac{1}{I(p, q)} = -\epsilon^2\frac{\nabla_p^2 I_0(p, q)}{I_{-1}^2} + \mathrm{O}(\epsilon^3) \tag{24}$$

with $\nabla_p^2 I_0(p, q) = \left[\nabla_p^2 I(p, q)\right]_{\epsilon=0}$. Analogous results with ∇_p^2 replaced by ∇_q^4 hold. Thus both $\eta_{L2}^{(1)}$ and $\eta_{L4}^{(1)}$ are indeed of order ϵ^2, and for their $\mathrm{O}(\epsilon^2)$ expansion coefficients $\eta_{L2,4}^{(1,2)}$ we obtain from equations (15), (16) and (24) the results

$$\eta_{L2}^{(1,2)} = -\frac{K_m}{8-m}\frac{1}{I_{-1}^2}\int_p^{(4-m/2)}\frac{1}{p^2+1}\nabla_p^2 I(p, 1)|_{\epsilon=0} \tag{25}$$

and

$$\eta_{L4}^{(1,2)} = -\frac{K_{4-m/2}}{2m(m+2)}\frac{1}{I_{-1}^2}\int_q^{(m)}\frac{1}{1+q^4}\nabla_q^4 I(1, q)|_{\epsilon=0}. \tag{26}$$

3. Epsilon expansions of $\eta_{L2}^{(1)}$ and $\eta_{L4}^{(1)}$

We are now ready to present the announced proof of consistency. We shall show that the ϵ expansions of the correlation exponents η_{L2} and η_{L4} are related to the coefficients $\eta_{L2}^{(1)}$ and $\eta_{L4}^{(1)}$ via

$$\eta_{L2,4} = \frac{n+2}{(n+8)^2}\eta_{L2,4}^{(1)} + \mathrm{O}(\epsilon^3). \tag{27}$$

Since the prefactor $(n+2)/(n+8)^2$ on the right-hand side reduces to $1/n$ in the large-n limit, consistency between the ϵ expansions to second order and the $1/n$ expansions to first order is an immediate consequence.

The $\mathrm{O}(\epsilon^2)$ results of [9, 10] for the exponents $\eta_{L2,4}$ involved single integrals, which for general m had to be computed by numerical means. In the notation of the second of these publications, they read

$$j_\phi(m) \equiv B_m\int_0^\infty \mathrm{d}\upsilon\, \upsilon^{m-1}\Phi^3(\upsilon; m, d^*) \tag{28}$$

and

$$j_\sigma(m) \equiv B_m \int_0^\infty \mathrm{d}\upsilon \upsilon^{m+3} \Phi^3(\upsilon; m, d^*), \tag{29}$$

where

$$B_m = \frac{S_{4-m/2} S_m}{I_{-1}^2} \tag{30}$$

and

$$\Phi(\upsilon; m, d) \equiv G^{(0)}(1, \upsilon). \tag{31}$$

The latter is the scaling function associated with the free position-space propagator $G^{(0)}(r, z)$, whose scaling properties

$$G^{(0)}(r, z) = r^{-2+\epsilon} G^{(0)}(1, zr^{-1/2}) = z^{-4+2\epsilon} G^{(0)}(rz^{-2}, 1) \tag{32}$$

we recall. For general values of m and d, it is a difference of two generalized hypergeometric functions ${}_1F_2$. This is why no analytic results for the integrals $j_\phi(m)$ and $j_\sigma(m)$ are available for general m. In [10], the reader may find numerical results for them at $m = 1, 2, \ldots, 7$ along with analytical ones for $m = 2$ and $m = 6$.

To prove the relations (27), we must show that

$$\eta_{L2}^{(1,2)} = \frac{2}{8-m} j_\phi(m) \tag{33}$$

and

$$\eta_{L4}^{(1,2)} = -\frac{1}{2m(m+2)} j_\sigma(m). \tag{34}$$

Let us start from equation (25). Its integral $\int_{\boldsymbol{p}}^{(4-m/2)}$ has the form of a scalar product $\langle f|g\rangle$ in $L_2(\mathbb{R}^{4-m/2})$, the space of square integrable functions, that is evaluated in the $\boldsymbol{p}$-representation. In $\boldsymbol{r}$-space the bra $\langle f|$ is represented by $f(r)^* \equiv \langle f|\boldsymbol{r}\rangle$, where $f(r)$ is the Fourier $\boldsymbol{q}$-transform of $G^{(0)}(r, z)$, taken at an arbitrary unit $\boldsymbol{q}$-vector $\hat{\boldsymbol{q}}$. Performing the angular integrations in the required m-dimensional integral is straightforward and yields

$$f(r) = \int \mathrm{d}^m z\, G^{(0)}(r, z)\, \mathrm{e}^{\mathrm{i}\hat{\boldsymbol{q}}\cdot\boldsymbol{z}} = (2\pi)^{m/2} \int_0^\infty \mathrm{d}z\, z^{m/2} J_{\frac{m}{2}-1}(z) G^{(0)}(r, z), \tag{35}$$

where from now on ϵ is set to zero in $G^{(0)}(r, z)$.

Likewise, $\langle \boldsymbol{r}|g\rangle \equiv g(r)$ is the Fourier $\boldsymbol{q}$-transform of the function $(-r^2)[G^{(0)}(r, z)]^2$ for $\boldsymbol{q} = \hat{\boldsymbol{q}}$. In the resulting expression for

$$\langle f|g\rangle = -(2\pi)^m \int \mathrm{d}^{4-m/2} r \int_0^\infty \mathrm{d}z \int_0^\infty \mathrm{d}z'\, r^2 (zz')^{m/2} J_{\frac{m}{2}-1}(z) G^{(0)}(r, z) J_{\frac{m}{2}-1}(z') [G^{(0)}(r, z')]^2 \tag{36}$$

we substitute the first of the scaling forms (32) along with equation (31). We then make the changes of variables $r \to \upsilon \equiv zr^{-1/2}$ in the radial part of the integration over $\boldsymbol{r}$ and $z' \to \zeta = z'/z$. The resulting integral over z is the special case of the closure relation for Bessel functions[6]

$$\int_0^\infty \mathrm{d}z\, z J_\mu(\zeta z) J_\mu(bz) = \delta(\zeta - b)/\zeta \tag{37}$$

[6] See, for instance, [19].

with $\mu = m/2 - 1$ and $b = 1$. The integral over ζ can now be performed. Substituting the result into equation (25) and noting (28) and (30) then gives the asserted result (33) for the coefficient $\eta_{L2}^{(1,2)}$.

The corresponding expression (34) for $\eta_{L4}^{(1,2)}$ can be proven in an analogous fashion. The integral $\int_{\boldsymbol{q}}^{(m)}$ is a scalar product $\langle h|w\rangle$ in $L_2(\mathbb{R}^m)$ between $h(z)$, the Fourier $\boldsymbol{p}$-transform of $G(r, z)$, and $w(z)$, that of $z^4[G^{(0)}(r, z)]^2$, taken at a unit $\boldsymbol{p}$-vector $\hat{\boldsymbol{p}}$. We perform the angular integrals in the Fourier integrals $\int \mathrm{d}^{4-m/2}r$ and $\int \mathrm{d}^{4-m/2}r'$, and make the changes of variables $z \to \upsilon = zr^{-1/2}$ and $r' \to \zeta = r'/r$. The integral over r is of the form (37) with $\mu = 1 - m/4$ and $b = 1$. Once the integral over ζ is performed, the desired result follows from equations (26), (30) and (29).

4. Concluding remarks

In this paper, we have shown that the large-n expansion yields $\mathrm{O}(1/n)$ results for the correlation exponents η_{L2} and η_{L4} and the related anisotropy exponent θ for general m, which are fully consistent with the ϵ-expansion of [9, 10]. In view of the long-standing discrepancies mentioned in the introduction and the great technical challenges encountered in both expansion methods beyond lowest order, the established consistency is very gratifying, providing nontrivial checks of the results of both expansions given in [9, 10, 13], respectively.

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